From Ruin Theory to Solvency in Non-Life Insurance

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Aim of this presentation

We start from Lundberg’s thesis (1903) on ruin theory and modify his model step by step until we arrive at today’s solvency considerations.
Cramér-Lundberg model

Consider the surplus process \((C_t)_{t \geq 0}\) given by

\[ C_t = c_0 + \pi t - \sum_{i=1}^{N_t} Y_i, \]

where

- \(c_0 \geq 0\) initial capital,
- \(\pi > 0\) premium rate,
- \(N_t \geq 0\) homogeneous compound Poisson claims process,

satisfying the net profit condition (NPC): \(\pi > \mathbb{E}[L_1]\).
Ultimate ruin probability

The ultimate ruin probability for initial capital $c_0 \geq 0$ is given by

$$\psi(c_0) = \mathbb{P}\left[ \inf_{t \in \mathbb{R}_+} C_t < 0 \mid C_0 = c_0 \right] = \mathbb{P}_{c_0}\left[ \inf_{t \in \mathbb{R}_+} C_t < 0 \right],$$

i.e. this is the infinite time horizon ruin probability.

Under (NPC):

$$\psi(c_0) < 1 \text{ for all } c_0 \geq 0.$$
Lundberg’s exponential bound

Assume (NPC) and that the Lundberg coefficient $\gamma > 0$ exists. Then, we have exponential bound

$$\psi(c_0) \leq \exp\{-\gamma c_0\},$$

for all $c_0 \geq 0$.

This is the light-tailed case, i.e. for the existence of $\gamma > 0$ we need exponentially decaying survival probabilities of the claim sizes $Y_i$. 
Subexponential case

Von Bahr, Veraverbeke, Embrechts investigate the heavy-tailed case.

In particular, for $Y_i \overset{i.i.d.}{\sim} \text{Pareto}(\alpha > 1)$ and (NPC):

$$\psi(c_0) \sim \text{const } c_0^{-\alpha+1} \quad \text{as } c_0 \to \infty.$$  

Heavy-tailed case provides a much slower decay.
Discrete time ruin considerations

Insurance companies cannot continuously control their surplus processes.

They close their books and check their surplus on a yearly time grid.

Consider the discrete time ruin probability

\[
\mathbb{P}_{c_0} \left( \inf_{n \in \mathbb{N}_0} C_n < 0 \right) \leq \mathbb{P}_{c_0} \left( \inf_{t \in \mathbb{R}_+} C_t < 0 \right) = \psi(c_0).
\]

This leads to the study of the random walk \((C_n - c_0)_{n \in \mathbb{N}_0}\) for (discrete) accounting years \(n \in \mathbb{N}_0\).
One-period ruin problem

Insured buy one-year non-life insurance contracts: why bother about ultimate ruin probabilities?

Moreover, initial capital $c_0 \geq 0$ needs to be re-adjusted every accounting year.

Consider the (discrete time) one-year ruin probability

$$
\mathbb{P}_{c_0} [C_1 < 0] \leq \mathbb{P}_{c_0} \left[ \inf_{n \in \mathbb{N}_0} C_n < 0 \right] \leq \mathbb{P}_{c_0} \left[ \inf_{t \in \mathbb{R}_+} C_t < 0 \right] = \psi(c_0).
$$

This leads to the study of the surplus $C_1 = c_0 + \pi - \sum_{i=1}^{N_1} Y_i$ at time 1.
One-period problem and real world considerations

Why do we study so complex models when the real world problem is so simple?

- Total asset value at time 1: $A_1 = c_0 + \pi$.
- Total liabilities at time 1: $L_1 = \sum_{i=1}^{N_1} Y_i$.

$$C_1 = c_0 + \pi - \sum_{i=1}^{N_1} Y_i = A_1 - L_1 \geq 0.$$  \hspace{1cm} (1)

There are many modeling issues hidden in (1)! We discuss them step by step.
Value-at-Risk (VaR) risk measure

\[ C_1 = A_1 - L_1 \geq 0. \]

- **Value-at-Risk on confidence level** \( p = 99.5\% \) (Solvency II):
  - choose \( c_0 \) minimal such that
  \[
  \mathbb{P}_{c_0} [C_1 \geq 0] = \mathbb{P} [A_1 \geq L_1] = \mathbb{P} [L_1 - c_0 - \pi \leq 0] \geq p.
  \]

- **Choose other (normalized) risk measures** \( \varrho : \mathcal{M} \subset L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R} \)
  and study
  \[
  \varrho (L_1 - A_1) = \varrho (L_1 - c_0 - \pi) \leq 0,
  \]
  where “\( \leq \)” implies **SOLVENCY** w.r.t. risk measure \( \varrho \).
Asset return and financial risk (1/2)

• Initial capital at time 0: $c_0 \geq 0$.

• Premium received at time 0 for accounting year 1: $\pi > 0$.

Total asset value at time 0: $a_0 = c_0 + \pi > 0$.

This asset value $a_0$ is invested in different assets $k \in \{1, \ldots, K\}$ at time 0.

asset classes
- cash and cash equivalents
- debt securities (bonds, loans, mortgages)
- real estate & property
- equity, private equity
- derivatives & hedge funds
- insurance & reinsurance assets
- other assets
Choose an asset portfolio $\mathbf{x} = (x_1, \ldots, x_K)' \in \mathbb{R}^K$ at time 0 with initial value $a_0 = \sum_{k=1}^{K} x_k S_0^{(k)}$.

where $S_t^{(k)}$ is the price of asset $k$ at time $t$. This provides value at time 1

$$A_1 = \sum_{k=1}^{K} x_k S_1^{(k)} = a_0 (1 + \mathbf{w}' \mathbf{R}_1),$$

for asset strategy $\mathbf{w} = \mathbf{w}(\mathbf{x}) \in \mathbb{R}^K$ and (random) return vector $\mathbf{R}_1$ at time 1.

$$\varrho(L_1 - A_1) = \varrho(L_1 - a_0 (1 + \mathbf{w}' \mathbf{R}_1)) \leq 0.$$

where “$\leq$” implies solvency w.r.t. risk measure $\varrho$ and business plan $(L_1, a_0, \mathbf{w})$. 
Insurance claim liability modeling

**MAIN ISSUE:** modeling of insurance claim \( L_1 = \sum_{i=1}^{N_1} Y_i \).

- Insurance claims are neither known nor can immediately be settled at occurrence!

- Insurance claims of accounting year 1 generate *insurance liability cash flows* \( X \):

\[
X = (X_1, X_2, \ldots) \quad \text{with} \quad X_t \text{ payment in accounting year } t.
\]

**Question:** How is the cash flow \( X \) related to the insurance claim \( L_1 \)?
Best-estimate reserves

Choose a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) with filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{N}_0}\) and assume \(X\) is \(\mathbb{F}\)-adapted.

1st attempt to define \(L_1\):

\[
L_1 = X_1 + \sum_{s \geq 2} P(1, s) \mathbb{E}[X_s | \mathcal{F}_1],
\]

where

- \(P(1, s)\) is the zero-coupon bond price at time 1 for maturity date \(s\);
- \(\mathbb{E}[X_s | \mathcal{F}_1]\) is the best-estimate reserve (prediction) of \(X_s\) at time 1.
1st attempt to define $L_1$

$$L_1 = X_1 + \sum_{s \geq 2} P(1, s) \mathbb{E} [X_s | F_1].$$

**Issue:** Solvency II asks for economic balance sheet, but $L_1$ is *not* an economic value.

(a) Risk margin is missing. Any risk-averse risk bearer asks for such a (profit) margin.

(b) Zero-coupon bond prices and claims cash flows $X_s$, $s \geq 2$, may be influenced by the same risk factors and, thus, *there is no decoupling* (2).
Choose an appropriate state-price deflator \( \varphi = (\varphi_t)_{t \geq 1} \) and

\[
L_1 = X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \mathbb{E}[\varphi_s X_s | \mathcal{F}_1].
\]

- \( \varphi = (\varphi_t)_{t \geq 1} \) is a strictly positive, a.s., and \( \mathbb{F} \)-adapted.
- \( \varphi = (\varphi_t)_{t \geq 1} \) reflects price formation at financial markets, in particular,

\[
P(1, s) = \frac{1}{\varphi_1} \mathbb{E}[\varphi_s | \mathcal{F}_1].
\]

- If \( \varphi_s \) and \( X_s \) are positively correlated, given \( \mathcal{F}_1 \), then

\[
L_1 \geq X_1 + \sum_{s \geq 2} P(1, s) \mathbb{E}[X_s | \mathcal{F}_1].
\]
Solvency at time 0

- Choose a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) such that it carries the random vectors \(\varphi\) (state-price deflator), \(R_1\) (returns of assets) and \(X\) (insurance liability cash flows) in a reasonable way.

- The business plan \((X, a_0, w)\) is solvent w.r.t. the risk measure \(\varrho\) and state-price deflator \(\varphi\) if

\[
\varrho(L_1 - A_1) = \varrho \left( X_1 + \sum_{s \geq 2} \frac{1}{\varphi_1} \mathbb{E} [\varphi_s X_s | \mathcal{F}_1] - a_0 (1 + w' R_1) \right) \leq 0.
\]

Thus, it is likely (measured by \(\varrho\) and \(\varphi\)) that the liabilities \(L_1\) are covered by assets \(A_1\) at time 1 in an economic balance sheet.
Summary of modeling tasks

• Provide reasonable stochastic models for $R_1$, $X$ and $\varphi$ (extrapolation).

• What is a reasonable profit margin for risk bearing expressed by $\varphi$?

• Which risk measure(s) $\varrho$ should be preferred? ($\Rightarrow$ No-acceptability arbitrage!)

• Modeling is often split into different risk modules:
  ★ (financial) market risk
  ★ insurance risk (underwriting and reserve risks)
  ★ credit risk
  ★ operational risk

▷ Issue: dependence modeling and aggregation of risk modules.

• Aggregation over different accounting years and lines of business?

P. Artzner
Dynamic considerations

Are we happy with the above considerations?

▷ Not entirely!

Liability run-off is a multi-period problem:

We also want sensible dynamic behavior.

This leads to the consideration of super-martingales.