Asset Demand and Ambiguity Aversion

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Introduction

- Axiomatization of ambiguity aversion in decision makers’ preferences and utilities has been one of the hottest topics in decision theory.
- There is now a growing literature to study its implications on optimal portfolios and asset prices.
- This paper belongs to this strand of literature.
- We prove a generalized mutual fund theorem and characterize optimal portfolios as the investor becomes extremely ambiguity-averse.
- As a corollary, we give a rigorous sense in which the $1/N$-portfolio is optimal.
Setup

- We use the smooth model of ambiguity aversion by Klibanoff, Marinacci, and Mukerji (2005, *Econometrica*).
- Unlike the utility functions of Gilboa and Schmeidler (1989, *JMathE*), the utility functions of KMM allow us to see what would happen to optimal portfolios when investors become more ambiguity averse but remain to be equally risk averse.
- A single consumption period with the CARA-Normal assumption, but also with ambiguous expected asset returns.
- Use the same model as the model of portfolio selection in Maccheroni, Marinacci, and Ruffino (2013, *Econometrica*).
First Result: Generalized Mutual Fund Theorem

- If there were no ambiguity, the mutual fund theorem would hold: every investor’s optimal portfolio of risky assets would be a positive multiple of the single mutual fund.
- With ambiguous asset returns, a single mutual fund is, in general, not sufficient to cater for all investors.
- However, each investor’s optimal portfolio can be represented as a linear combination of some $K$ vectors, where $1 \leq K \leq N$. 
Second Result: Decomposition of Optimal Portfolios

- An expected-utility maximizer’s optimal portfolios can be decomposed into two parts:
  1. Vanishes as the degree of ambiguity aversion goes to infinity.
  2. Remains even when the degree of ambiguity aversion goes to infinity.

- Useful in the analysis of a factor model, in which a small number of factors determines the asset returns.
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Assets and Ambiguity

- $N$ risky assets with gross returns $X$ and a riskless asset with $R$.
- Ambiguity is represented by an $N$-dimensional random vector $M$.
- Assume that

$$
\begin{pmatrix}
M \\
X
\end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix}
\mu_M \\
\mu_M
\end{pmatrix}, \begin{pmatrix}
\Sigma_M & \Sigma_M \\
\Sigma_M & \Sigma_X
\end{pmatrix} \right),
$$

Then $X|M \sim \mathcal{N}(M, \Sigma_X - \Sigma_M)$.
- Each realization of $M$ specifies a model, and the investor is faced with model uncertainty on expected returns.
Examples of $\mu_M$, $\Sigma_X$, and $\Sigma_M$

- Take the Bayesian view and stretch the meaning of the “principle of insufficient reason.”

  $\mu_M = \delta 1$ and $\Sigma_M = \begin{pmatrix} \sigma^2 & \kappa \\ \kappa & \sigma^2 \end{pmatrix}$.

- Imagine that asset prices were observed during the last $T$ periods and take the frequentialist’s view.

  $\Sigma_M = \frac{1}{T} \Sigma_X$. 
Utility Functions

- Utility function over random consumptions $Z$:

$$U_{\gamma, \theta}(Z) = E \left[ u_\gamma \left( u_\theta^{-1} \left( E \left[ u_\theta(Z) | M \right] \right) \right) \right],$$

where $u_\theta(z) = -\exp(-\theta z)$ and $u_\gamma(z) = -\exp(-\gamma z)$.

- If $\theta = \gamma$, then $U_{\gamma, \theta}$ is a CARA expected utility function.

- If $Z = a^\top X + bR$ for some portfolio $(a, b) \in \mathbb{R}^N \times \mathbb{R}$, then $U_{\gamma, \theta}(Z) = -\exp(-\gamma V_{\gamma, \theta}(a, b))$, where

$$V_{\gamma, \theta}(a, b) = \mu_M^\top a + Rb - \frac{\gamma}{2} a^\top \Sigma_M a - \frac{\theta}{2} a^\top (\Sigma_X - \Sigma_M) a.$$

This is a mean-variance utility function of MMR.
Portfolio Choice

- Initial wealth $W \in \mathbb{R}$
- The utility maximization problem is

$$\max_{(a,b) \in \mathbb{R}^N \times \mathbb{R}} U_{\gamma,\theta}(a^\top X + bR)$$

subject to $1^\top a + b \leq W$.

- The optimal Portfolio is

$$a = (\gamma \Sigma_M + \theta (\Sigma_X - \Sigma_M))^{-1}(\mu_M - R1).$$

If $\theta = \gamma$, then

$$a = \frac{1}{\gamma} \Sigma_X^{-1}(\mu_M - R1).$$
Define

\[ \eta \equiv \frac{\gamma}{\theta} - 1 \quad \text{and} \quad Q \equiv \Sigma_X^{-1} \Sigma_M. \]

- \( \eta \) represents the degree of ambiguity aversion in excess of risk aversion.
- \( Q \) is roughly equal to the ratio of the variance of asset returns due solely to ambiguity to the total variance of these asset returns.

Define \( \alpha : (-1, \infty) \to \mathbb{R}^N \) by

\[ \alpha(\eta) = (I_N + \eta Q)^{-1} \Sigma_X^{-1} (\mu_M - R1) \]

Then the optimal portfolio \( \alpha \) is equal to \( \theta^{-1} \alpha(\eta) \).
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Theorem (Generalized Mutual Fund Theorem)

If $\mu_M - R 1 \neq 0$, then there are a $K \in \{1, 2, \ldots, N\}$ and $K$ eigenvectors $v_1, v_2, \ldots, v_K$ of $Q$ with corresponding non-negative eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_K$ such that

$$
\alpha(\eta) = \sum_{k=1}^{K} \frac{1}{1 + \lambda_k \eta} v_k.
$$
Proof of the Generalized Mutual Fund Theorem

1. The variance-ratio matrix $Q$ need not be symmetric, but there is a basis $(w_1, w_2, \ldots, w_N)$ of $\mathbb{R}^N$ with each $w_n$ an eigenvector of $Q$.
2. There is a $(c_1, c_2, \ldots, c_N) \in \mathbb{R}^N$ such that $\alpha(0) = \sum_{n=1}^{N} c_n w_n$.
3. Partition the set $\{c_1 w_1, c_2 w_2, \ldots, c_n w_n\}$ by the corresponding eigenvalues and consider the sums of the form $\sum_n c_n w_n$ where the sum is taken over each subset of a common eigenvalue.
4. Denote these sums by $v_1, v_2, \ldots, v_K$ and order them with the increasing order of the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_K$.
5. Since $\alpha(\eta) = (I_N + \eta Q)^{-1} \alpha(0)$ and each $v_k$ is an eigenvector of $(I_N + \eta Q)^{-1}$ with eigenvalue $(1 + \eta \lambda_k)^{-1}$, the theorem is established.
Implications of the Generalized Mutual Fund Theorem

- The $K$ mutual funds $v_1, v_2, \ldots, v_K$ can cater for all investors, regardless of the values of $\eta$, who believe $\mathcal{N}(\mu_M, \Sigma_M)$.

- If $\lambda_k > 0$, then the demand for $v_k$ converges to zero as $\eta \to \infty$, though the speed of convergence depends on the value of $\lambda_k$.

- If $\lambda_1 = 0$, then the demand for $v_1$ does not depend on $\eta$. This is because $v_1 \in \text{Ker} \, Q = \text{Ker} \, \Sigma_M$ and $v_1$ involves no ambiguity.

- If $\Sigma_M = \lambda \Sigma_X$, then $Q = \lambda I_N$. Hence $\alpha(0)$ is an eigenvector of $Q$. Thus $K = 1$ and the original mutual fund theorem holds.
Application 1: Heterogeneous Investors

There are $I$ investors $i = 1, 2, \ldots, I$ having utility functions $U_{\gamma^i, \theta^i}$. Let

$$
\eta^i = \gamma^i / \theta^i - 1
$$

Proposition

Suppose that $\eta^1 = \eta^2 = \cdots = \eta^I$. Then:

1. The demands $a^i$ for risky assets are positive multiples of one another.

2. Denote the common value of the $\eta^i$ by $\eta$. Define $\theta$ and $\gamma$ by

$$
\theta^{-1} = \sum_i (\theta^i)^{-1} \text{ and } \eta = \gamma / \theta - 1.
$$

Then $\theta^{-1} \alpha(\eta) = \sum_i a_i$.

▶ A single mutual fund is sufficient for all investors.
▶ The representative investor’s risk tolerance is equal to the sum of the investors’ risk tolerances, and the degree of extra ambiguity is equal to each individual investor’s counterpart.
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Decomposition of Optimal Portfolios

When it is possible to form ambiguity-free portfolios, as in models of the home bias puzzle, the following theorem is useful.

**Theorem (Decomposition of Optimal Portfolios)**

1. *There are two negative semidefinite matrices* $\Sigma_1$ *and* $\Sigma_2$ *such that* $\Sigma_1 + \Sigma_2 = \Sigma_X$, *$\mathbb{R}^N$ is the direct sum of* Row $\Sigma_1$ *and* Row $\Sigma_2$, *and* Row $\Sigma_1 = $ Row $\Sigma_M$.

2. *For each such* $(\Sigma_1, \Sigma_2)$, *there exist a unique* $(w_1, w_2) \in $ Row $\Sigma_1 \times$ Row $\Sigma_2$ *such that* $\mu_M - R1 = w_1 + w_2$, *and a unique* $(v_1, v_2) \in $ Ker $\Sigma_1 \times$ Ker $\Sigma_2$ *such that* $\Sigma_2 v_1 = w_2$ *and* $\Sigma_1 v_2 = w_1$.

3. *For each such* $(v_1, v_2)$, $\alpha(0) = v_1 + v_2$ *and* $\alpha(\eta) \rightarrow v_1$ *as* $\eta \rightarrow \infty$. 
Implications of the Decomposition Theorem

- In symbols,

\[
\alpha(0) = \Sigma_X^{-1}(\mu_M - R1)
\]

\[
\iff \Sigma_X \alpha(0) = \mu_M - R1
\]

\[
\iff (\Sigma_1 + \Sigma_2)(v_1 + v_2) = w_1 + w_2
\]

\[
\iff \Sigma_1 v_2 + \Sigma_2 v_1 = w_1 + w_2
\]

(by this theorem) \iff \Sigma_1 v_2 = w_1 \text{ and } \Sigma_2 v_1 = w_2.

- An expected-utility-maximizing investor’s portfolio \( \alpha(0) \) can be decomposed into two portfolios \( v_1 \) and \( v_2 \). \( v_1 \) involves no ambiguity.

- If \( \Sigma_M \) is positive definite, then \( v_1 = 0 \) and \( \alpha(\eta) \to 0 \) as \( \eta \to \infty \).
Application 2: Asymptotic Portfolio Weights

If $\Sigma_M$ is positive definite, then $\alpha(\eta) \to 0$ as $\eta \to \infty$.

How about the portfolio weights?

Proposition

*If $\Sigma_M$ is positive definite and $\mathbf{1}^\top \alpha(\eta) > 0$ for every sufficiently large $\eta$, then, as $\eta \to \infty$,*

$$
\frac{1}{\mathbf{1}^\top \alpha(\eta)} \alpha(\eta) \to \frac{1}{\mathbf{1}^\top \Sigma_M^{-1}(\mu_M - R\mathbf{1})} \Sigma_M^{-1}(\mu_M - R\mathbf{1}).
$$
Implications of the Asymptotics

- The portfolio weight of an extremely ambiguity averse investor is similar to the portfolio weight of the expected-utility CARA with mean vector $\mu_M$ and covariance matrix $\Sigma_M$.
- The relevant covariance matrix is not $\Sigma_X$ but $\Sigma_M$.
- If, at the same time, $\theta \to 0$ so that
  \[
  \frac{1^T \alpha(\eta)}{\theta} \to c \in R_{++},
  \]
  then the demands for risky assets do not vanish and has weights $\Sigma_M^{-1}(\mu_M - R1)$ in the limit:
  \[
  a = \theta^{-1} \alpha(\eta) \to c \Sigma_M^{-1}(\mu_M - R1).
  \]
Application 3: $1/N$-portfolio

Proposition

Suppose that there is a $\delta \in \mathbb{R}$ with $\delta \neq \mathbb{R}$ such that $\mu_M = \delta 1$, and there are a $\sigma \in \mathbb{R}_{++}$ and a $\kappa \in \mathbb{R}$ such that $-(N-1)^{-1} < \kappa / \sigma^2 < 1$ and

$$\Sigma_M = \begin{pmatrix} \sigma^2 & \kappa \\ \kappa & \sigma^2 \end{pmatrix}.$$ 

Then, as $\eta \to \infty$,

$$\frac{1}{\alpha(\eta)^\top \beta(\eta)} \to \frac{1}{N} 1.$$

If, at the same time, $\theta \to 0$ so that $\theta^{-1} 1^\top \alpha(\eta) \to c \in \mathbb{R}_{++}$, then

$$a = \frac{1}{\theta} \alpha(\eta) \to \frac{c}{N} 1.$$
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If there is a \( \lambda \in [0, 1] \) such that \( \Sigma_M = \lambda \Sigma_X \), then

\[
\alpha(\eta) = \frac{1}{1 + \lambda \eta} \Sigma_X^{-1} (\mu_M - R1)
\]

for every \( \eta > -1 \).

Corollary

For every \( n \geq 1 \), if \( \alpha_n(\eta) > 0 \) for some \( \eta \), then \( \alpha_n(\eta) \downarrow 0 \) as \( \eta \uparrow \infty \).
Proposition

If there is a \( \lambda \in (0, 1] \) such that \( \lambda \Sigma_X - \Sigma_M \) is positive semidefinite and rank \( \Sigma_M + \text{rank} (\lambda \Sigma_X - \Sigma_M) = N \), then there are a \( v_1 \) and a \( v_2 \) such that

\[
\alpha(\eta) = v_1 + \frac{1}{1 + \lambda \eta} v_2.
\]

for every \( \eta > -1 \).

Corollary (Generalization of Proposition 8 of MMR)

Suppose in addition that there are an \( L < N \) and a \( \hat{\Sigma}_M \in S^L_+ \) such that

\[
\Sigma_M = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Sigma}_M \end{pmatrix},
\]

then, for every \( n > L \), if \( \alpha_n(\eta) > 0 \) for some \( \eta \), then \( \alpha_n(\eta) \downarrow 0 \) as \( \eta \uparrow \infty \).
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Let $L < N$ and

$$X = \beta^\top Y + Z,$$

where $\beta \in \mathbb{R}^{L \times N}$ and

$$
\begin{pmatrix}
  G \\
  Y \\
  H \\
Z
\end{pmatrix}
\sim
\mathcal{N}
\begin{pmatrix}
  \begin{pmatrix}
    \mu_G \\
    \mu_G \\
    \mu_H \\
    \mu_H
  \end{pmatrix},
  \begin{pmatrix}
    \Sigma_G & \Sigma_G & \Sigma_{GH} & \Sigma_{GH} \\
    \Sigma_G & \Sigma_Y & \Sigma_{GH} & 0 \\
    \Sigma_{HG} & \Sigma_{HG} & \Sigma_H & \Sigma_H \\
    \Sigma_{HG} & 0 & \Sigma_H & \Sigma_Z
  \end{pmatrix}
\end{pmatrix}.
$$

Then,

$$
\begin{pmatrix}
  Y \\
Z
\end{pmatrix}
\mid
\begin{pmatrix}
  G \\
  H
\end{pmatrix}
\sim
\mathcal{N}
\begin{pmatrix}
  \begin{pmatrix}
    G \\
    H
  \end{pmatrix},
  \begin{pmatrix}
    \Sigma_Y - \Sigma_G & -\Sigma_{GH} \\
    -\Sigma_{HG} & \Sigma_Z - \Sigma_H
  \end{pmatrix}
\end{pmatrix}.
$$

If $\text{rank}\beta = L$ and $\mathbb{R}^N = \text{Ker} \beta + \text{Ker} \Sigma_Z$, then there is a nice way to characterize $\alpha(0) = \nu_1 + \nu_2$. 
Optimal Portfolios in the Factor Model

We further assume that

$$\beta = \begin{pmatrix} I_L & \hat{\beta} \end{pmatrix} \quad \text{and} \quad \Sigma_Z = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Sigma}_Z \end{pmatrix}. $$

- If $\Sigma_G$ is positive definite and $\Sigma_H = 0$, then

$$v_1 = \begin{pmatrix} -\hat{\beta} \hat{\Sigma}^{-1}_Z \left( R\hat{\beta}^\top 1_L - R 1_{N-L} \right) \\ \hat{\Sigma}_Z^{-1} \left( R\hat{\beta}^\top 1_L - R 1_{N-L} \right) \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} \Sigma_Y^{-1} (\mu_Y - R 1_L) \\ 0 \end{pmatrix}. $$

- If $\Sigma_G = 0$ and $\Sigma_M$ is positive definite, then

$$v_1 = \begin{pmatrix} \Sigma_Y^{-1} (\mu_Y - R 1_L) \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -\hat{\beta} \hat{\Sigma}^{-1}_Z \left( R\hat{\beta}^\top 1_L - R 1_{N-L} \right) \\ \hat{\Sigma}_Z^{-1} \left( R\hat{\beta}^\top 1_L - R 1_{N-L} \right) \end{pmatrix}. $$
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- Studied the CARA-Normal setup with ambiguity.
- Provided a generalized mutual fund theorem.
- Decomposed each optimal portfolio into two parts: as the investor becomes unboundedly ambiguity averse, the first part remains but the second part vanishes.
- Should accommodate CRRA utility functions and dynamic trading, as in Ju and Miao (2012, *Econometrica*).
- Should extend the Bayesian portfolio analysis, as in Avramov and Zhou (2010, *Annual Review of Financial Economics*), and the principal component analysis to the case with ambiguity.