Financial Markets: Behavioral Equilibrium and Evolutionary Dynamics

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November 21rst 2013/ Swiss-Kyoto Symposium
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Our Goal

New modeling of financial markets based on observables like investors’ wealth and investment decisions and not on expectations and utility functions in order to derive a few basic principles that long-term investors should follow when they invest in competition with other investors and the investment results are subject to exogeneous and endogenous uncertainty.
Fundamental questions to be answered:

a) “What is the best strategy for playing the stock market? Should one concentrate on **fundamentals** or should one focus on the **psychology of the market?**” (Cass and Shell (JPE, 1981)).

b) Should one follow an **active strategy** or is **passive investing** the best rule?

c) If passive is best is there a way to select the assets cleverly or shall one invest in the **market portfolio**?

d) If active strategies are superior shall they be based on **expectations** and **optimization of objective functions**?
Model Requirements

Since we consider long-term investors we need a model with many time periods.
Since investment results are uncertain we need to allow for exogenous and endogeneous uncertainty.
Since investors compete for market capital one should model the market interaction.
One should allow any degree of rationality of the competing strategies. Otherwise one falls into the trap called market risk.
Model Requirements

Note that these requirements rule out famous models like e.g. macroeconomic models with a representative agent (DSGE). the Capital Asset Pricing Model (CAPM). the General Equilibrium Model with time and uncertainty (GEI).

The model we seek shall be based on observables and must have elements of dynamic stochastic games “played” on many asset markets simultaneously and possibly with behavioural participants.

Key idea: Instead of using the general equilibrium solution concept of equilibria in plans, prices and price expectations or the game theoretic solution concept of a Nash equilibrium we apply evolutionary solutions concepts like survival and evolutionary stability.
The Basic Model

**Randomness.** $S$ a measurable space of "states of the world"
$s_t \in S \ (t = 1, 2, ...) \ \text{state of the world at date } t;$
$s_1, s_2, ... \ \text{an exogenous stochastic process}.$

**Assets.** There are $K > 1$ assets.

**Dividends.** At each date $t$, assets $k = 1, ..., K$ pay dividends
$D_{t,k}(s^t) \geq 0, \ k = 1, ..., K,$ depending on the history

$$s^t := (s_1, ..., s_t)$$

of the states of the world up to date $t.$
Assumptions on dividends.

\[ \sum_{k=1}^{K} D_{t,k}(s_t^t) > 0; \quad ED_{t,k}(s_t^t) > 0, \quad k = 1, \ldots, K, \quad t = 1, 2, \ldots, \]

where \( E \) is the expectation with respect to the underlying probability \( P \).

**Asset supply.** Total mass (the number of ”physical units”) of asset \( k \) available at each date \( t \) is \( V_k > 0 \).
Investors and their portfolios.
There are \( N \) investors (traders) \( i \in \{1, \ldots, N\} \).
Investor \( i \) at date \( t = 0, 1, 2, \ldots \) selects a portfolio

\[
x^i_t = (x^i_{t,1}, \ldots, x^i_{t,K}),
\]

where \( x^i_{t,k} \) is the number of units of asset \( k \) in the portfolio \( x^i_t \).
The portfolio \( x^i_t \) for \( t \geq 1 \) depends, generally, on the current and previous states of the world:

\[
x^i_t = x^i_t(s^t), \quad s^t = (s_1, \ldots, s_t).
\]
Asset prices. We denote by $p_t \in \mathbb{R}^K_+$ the vector of market prices of the assets. For each $k = 1, \ldots, K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, \ldots, p_{t,K})$ stands for the price of one unit of asset $k$ at date $t$. The prices might depend on the current and previous states of the world:

$$p_{t,k} = p_{t,k}(s^t), \quad s^t = (s_1, \ldots, s_t).$$

The scalar product

$$\langle p_t, x^i_t \rangle := \sum_{k=1}^K p_{t,k}x^i_{t,k}$$

expresses the market value of the investor $i$’s portfolio $x^i_t$ at date $t$. 
The state of the market at date $t$:

$$(p_t, x^1_t, \ldots, x^N_t),$$

where $p_t$ is the vector of asset prices and $x^1_t, \ldots, x^N_t$ are the portfolios of the investors.

Investors’ budgets. At date $t = 0$ investors have initial endowments $w^i_0 > 0$ ($i = 1, 2, \ldots, N$). Trader $i$’s budget (wealth) at date $t \geq 1$ is

$$w^i_t(p_t, x^i_{t-1}) := \langle D_t + p_t, x^i_{t-1} \rangle,$$

where

$$D_t(s^t) := (D_{t,1}(s^t), \ldots, D_{t,K}(s^t)).$$

Two components:

- the dividends $\langle D_t(s^t), x^i_{t-1} \rangle$ paid by the yesterday’s portfolio $x^i_{t-1}$;
- the market value $\langle p_t, x^i_{t-1} \rangle$ of the portfolio $x^i_{t-1}$ in the today’s prices $p_t$. 

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**Investment rate.** A fraction $\alpha$ of the budget is invested into assets. We will assume that the *investment rate* $\alpha \in (0, 1)$ is the same for all the traders. $\alpha$ may depend on $t$ and $s^t$.

**Investment proportions.** For each $t \geq 0$, each trader $i = 1, 2, \ldots, N$ selects a vector of *investment proportions* 

$$\lambda^i_t = (\lambda^i_{t,1}, \ldots, \lambda^i_{t,K}) \in \Delta^K$$

in the unit simplex $\Delta^K$, according to which the budget is distributed between assets.
Game-theoretic framework. We regard the investors $i = 1, 2, ..., N$ as players in an $N$-person stochastic dynamic game. The vectors of investment proportions $\lambda_t^i$ are the players’ actions or decisions.

Players’ decisions might depend on the history $s^t := (s_1, ..., s_t)$ of states of the world and the market history

$$H^{t-1} := (p^{t-1}, x^{t-1}, \lambda^{t-1}),$$

where

$$p^{t-1} := (p_0, ..., p_{t-1}),$$

$$x^{t-1} := (x_0, x_1, ..., x_{t-1}), \quad x_l = (x_1^l, ..., x_N^l),$$

$$\lambda^{t-1} := (\lambda_0, \lambda_1, ..., \lambda_{t-1}), \quad \lambda_l = (\lambda_1^l, ..., \lambda_N^l).$$
**Investment strategies.** A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions with values in $\Delta^K$

$$\Lambda_t^i(s^t, H^{t-1}), \ t = 1, 2, \ldots,$$

form an *investment strategy (portfolio rule)* $\Lambda^i$ of investor $i$.

**Basic strategies.** Strategies for which $\Lambda_t^i$ depends only on $s^t$, and not on the market history $H^{t-1} = (p^{t-1}, x^{t-1}, \lambda^{t-1})$. We will call such portfolio rules *basic*.

**Simple strategies.** Strategies for which $\Lambda_t^i$ is constant are so called fixed-mix (constant proportions) strategies. We will call such portfolio rules *simple*.
**Investor $i$’s demand function.** Given a vector of investment proportions $\lambda^i_t = (\lambda^i_{t,1}, ..., \lambda^i_{t,K})$ of investor $i$, the $i$’s demand function is

$$X^i_{t,k}(p_t, x^i_{t-1}) = \frac{\alpha \lambda^i_{t,k} w^i_t(p_t, x^i_{t-1})}{p_{t,k}}.$$  

where $\alpha$ is the investment rate.

**Short-run (temporary) equilibrium.** For each $t$, aggregate demand for every asset is equal to supply:

$$\sum_{i=1}^{N} X^i_{t,k}(p_t, x^i_{t-1}) = V_k, \quad k = 1, ..., K.$$  

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Equilibrium Market Dynamics

Prices:

\[ p_{t,k} V_k = \sum_{i=1}^{N} \alpha \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle, \quad k = 1, \ldots, K. \]

Portfolios:

\[ x_{t,k}^i = \frac{\alpha \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{p_{t,k}}, \quad k = 1, \ldots, K, \quad i = 1, 2, \ldots, N. \]

The vectors of investment proportions \( \lambda^i_t = (\lambda_{t,k}^i) \) are recursively determined by the investment strategies

\[ \lambda^i_t(s^t) := \Lambda^i_t(s^t, H^{t-1}), \quad i = 1, 2, \ldots, N. \]

Under mild assumptions on the strategy profile, the pricing equation has a unique solution \( p_t > 0 \).
Random Dynamical System (I)

Denote by

\[ r_t = (r_1^t, \ldots, r_N^t) \]

the random vector of the market shares

\[ r_i^t = \frac{w_i^t}{w_1^t + \ldots + w_N^t} \]

of the \( N \) investors. The dynamics of the vectors of market shares \( r_t \) is governed by the random dynamical system:

\[ r_{t+1}^i = \sum_{k=1}^{K} \left[ \alpha \langle \lambda_{t+1,k}, r_{t+1} \rangle + (1 - \alpha) \frac{D_{t+1,k} + p_{t+1,k}}{p_{t,k}} \right] \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle}, \]

\[ i = 1, \ldots, N, t \geq 0. \]
Random Dynamical System (II)

**Relative dividends.** Define the *relative dividends* of the assets $k = 1, \ldots, K$ by

$$d_{t,k} = d_{t,k}(s^t) := \frac{D_{t,k}(s^t)V_k}{\sum_{m=1}^{K} D_{t,m}(s^t)V_m}, \quad k = 1, \ldots, K, \quad t \geq 1,$$

and put $d_t(s^t) = (d_{t,1}(s^t), \ldots, d_{t,K}(s^t))$. The random dynamical system $r_t$ can then be explicitly written as

$$r_{t+1} = (1 - \alpha) \left[ Id - \left[ \frac{\lambda_i^{t,k}r_i^{t}}{\langle \lambda_t,k, r_t \rangle} \right]_{i}^{k} \right] \Lambda_{t+1}^{-1} \left[ \sum_{k=1}^{K} d_{t+1,k} \frac{\lambda_i^{t,k}r_i^{t}}{\langle \lambda_t,k, r_t \rangle} \right]_{i}^{k}$$

Nonlinear, defined in terms of rational functions (ratios of polynomials) with $N$ variables.
Marshallian temporary equilibrium.
We use the Marshallian “moving equilibrium method,” to model the dynamics of the asset market as sequence of consecutive temporary equilibria.
To employ this method one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set of vectors of investment proportions) can be temporarily fixed, while the other (in our case, the asset prices) can be assumed to rapidly reach the unique state of partial equilibrium.
Survival Strategies

**Survival strategies.** Given a strategy profile \((\Lambda^1, \ldots, \Lambda^N)\), we say that the portfolio rule \(\Lambda^1\) (or the investor 1 using it) *survives* with probability one if

\[
\inf_{t \geq 0} r^1_t > 0 \text{ (a.s.)},
\]

(the market share of investor 1 is bounded away from zero a.s. by a strictly positive random variable).

**Definition.** A portfolio rule is called a *survival strategy* if the investor using it survives with probability one (*irrespective of what portfolio rules are used by the other investors!*).

Our central goal is to identify survival strategies.
Definition of the survival strategy $\Lambda^*$. Put

$$\alpha_l = \alpha^{l-1}(1 - \alpha).$$

Define

$$\lambda_t^*(s^t) = (\lambda_{t,1}^*(s^t), \ldots, \lambda_{t,K}^*(s^t)), $$

where

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \alpha_l d_{t+l,k}.$$

Here, $E_t(\cdot) = E(\cdot | s^t)$ stands for the conditional expectation given $s^t$; $E_0(\cdot)$ is the unconditional expectation $E(\cdot)$.

Assume $\lambda_{t,k}^* > 0$ (a.s.); all $t$ and $k$.

The central results are as follows.
Theorem 1. *The portfolio rule $\Lambda^*$ is a survival strategy.*

We emphasize that the strategy $\Lambda^*$ is basic, and it survives in competition with any (not necessarily basic) strategies!

In the class of *basic* strategies, the survival strategy $\Lambda^*$ is *asymptotically unique*:

**Theorem 2.** *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$
\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.)}.
$$

**Theorem 3.** No strategy sufficiently distinct to the portfolio rule $\Lambda^*$ can survive in competition with $\Lambda^*$. 
The meaning of $\Lambda^*$. The portfolio rule $\Lambda^*$ defined by

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \alpha_l d_{t+l,k},$$

combines four general investment principles.

a) $\Lambda^*$ is purely based on **fundamental values** – the expectations of the flows of the discounted future dividends.

b) $\Lambda^*$ is **semi active** as it amounts to keep the investment proportions in line with the discounted future dividends.

c) The strategy $\Lambda^*$ is **completely diversified** analogously to the market portfolio which however is also based on price fluctuations.

d) In general the portfolio rule $\Lambda^*$ cannot be obtained from the optimization of some utility based on price expectations. But in some special cases it reduces to the **Kelly portfolio rule** prescribing to maximize the expected logarithm of the portfolio return – see below.
The i.i.d. case. If $s_t \in S$ are independent and identically distributed (i.i.d.) and

$$d_{t,k}(s^t) = d_k(s_t),$$

then

$$\lambda^*_{t,k} = \lambda^*_k = Ed_k(s_t),$$

does not depend on $t$, and so $\Lambda^*$ is a simple strategy, i.e. a fixed-mix (constant proportions) strategy. It is independent of the investment rate $\alpha$!

In the case of Arrow securities ("horse race model"), the expectations $ER_k(s_t)$ are equal to the probabilities of the states of the world ("betting your beliefs"). This is the Kelly portfolio rule maximizing the expected log returns.
Global Evolutionary Stability of $\Lambda^*$

Consider the i.i.d. case in more detail. It is important for quantitative applications and admits a deeper analysis of the model. Let us concentrate on fixed-mix strategies. In the class of such strategies, $\Lambda^*$ is globally evolutionarily stable:

**Theorem 4.** If among the $N$ investors, there is a group using $\Lambda^*$, then those who use $\Lambda^*$ survive, while all the others are driven out of the market (their market shares tend to zero a.s.).
Evolutionary Game-Theoretic Aspects

Synthesis of evolutionary and dynamic games

The notion of a survival strategy is the solution concept we adopt in the analysis of the market game.

This is a solution concept of a purely evolutionary nature.

No utility maximization or Nash equilibrium is involved.

On the other hand, the strategic framework we consider is the one characteristic for stochastic dynamic games (Shapley 1953).
In Order To Survive You Have To Win!

**Equivalence of Survival and Unbeatable Strategies** One might think that the focus on survival substantially restricts the scope of the analysis: ”one should care of survival only if things go wrong”.

It turns out, however, that the class of survival strategies coincides with the class of **unbeatable** strategies performing not worse in the long run – **in terms of wealth accumulation** – than any other strategies competing in the market.

Thus,

**in order to survive you have to win!**
Winning (=unbeatable) strategies of capital accumulation

Definition. A strategy $\Lambda$ is called unbeatable if it has the following property:

Suppose investor $i$ uses the strategy $\Lambda$, while all the others $j \neq i$ use any strategies. Then the wealth process $w^j_t$ of every investor $j \neq i$ cannot grow asymptotically faster than the wealth process $w^i_t$ of investor $i$: $w^j_t \leq H w^i_t$ (a.s.) for some random constant $H$.

It is quite easy to show that a strategy is a survival strategy if and only if it is unbeatable.
Pre-von Neumann / Pre-Nash game theory

The notion of a winning or unbeatable strategy was a central solution concept in the pre-von Neumann and pre-Nash game theory (as a branch of mathematics, pioneered by Bouton, Zermelo, Borel, 1900s - 1920s).

The question of determinacy of a game (existence of a winning strategy for one of the players) was among the key topics in game theory until 1950s. Dynamic games of complete information: Gale, Stewart, Martin ("Martin’s axiom").

The first mathematical paper in game theory ”solving” a game (= finding a winning strategy for one of the players) was:

Unbeatable strategies and evolutionary game theory

The basic solution concepts in evolutionary game theory – evolutionary stable strategies (Maynard Smith & Price, Schaffer) – may be regarded as “conditionally” unbeatable strategies (the number of mutants is small enough, or they are identical).

REFERENCES

The model described was developed in
I.V. Evstigneev, T. Hens, K.R. Schenk-Hoppé, Evolutionary stable stock markets, Economic Theory (2006);
The most general results:
Versions of the model

Short-lived assets

Assets live one period, yield payoffs, and then are identically reborn at the beginning of the next period.


The 2002 paper was inspired by

Model with a riskless asset

Handbook


Survey on Evolutionary Finance

Annals of Finance
Editor-in-Chief: Anne Villamil

Special Issue

Behavioral and Evolutionary Finance

May 2013

Guest Editors: Igor Evstigneev, Klaus R. Schenk-Hoppé and William T. Ziemba