When roll-overs do not qualify as numéraire: bond markets beyond short rate paradigms

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Introduction

Numéraires, Bubbles and Liquidity

Market models for bond markets

Bond markets as large financial markets

Existence of a bank account
Most of the term structure models in the literature are based on the fundamental assumption that bond prices $P(t, T)$ together with a *numéraire bank account process* $B(t)$ form an arbitrage-free market. Formally speaking this means that we can find an equivalent local martingale measure for the collection of stochastic processes $(B(t)^{-1}P(t, T))_{0 \leq t \leq T}$ representing bond prices discounted by the bank account’s current value. If we assume additionally that those local martingales are indeed martingales, then we arrive at the famous relationship

$$P(t, T) = E_Q \left[ \frac{B_t}{B_T} | \mathcal{F}_t \right]$$

for $0 \leq t \leq T$. If we assume alternatively the existence of forward rates, we arrive at the Heath-Jarrow-Morton (HJM) drift condition for the stochastic forward rate process encoding the previous local martingale property.
Goal of this work

- Bond market theory without assumptions on the existence of a bank account numéraire.
- Existence of martingale measures as a consequence of trading arguments.
- Analysis whether bank account numéraires exist and what they could mean.
- Relationship of numéraires and bubbles.
Economic definition of a numéraire

A positive portfolio in a financial market is considered a numéraire if arbitrary positive and negative (sic!) quantities of this portfolio may be held at any time by the investor.

This means that we always find counterparties for going short in arbitrary quantities. There are two reasons for counterparties not to accept such a deal: first, the belief that the numéraire is overpriced and hence buying it is not a good idea, or the belief that the numéraire default is close.
Consider a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)\), where the filtration satisfies the usual conditions. The price process of traded assets \((X_t)_{t \in [0, T]} = (X^0_t, \ldots, X^d_t)_{t \in [0, T]}\) is a \(d + 1\)-dimensional adapted process with càdlàg trajectories, where at least one process, say \(X^0\), is positive, i.e. \(X^0 > 0\). We introduce the process of discounted assets,

\[
S := (1, \frac{X^1}{X^0}, \ldots, \frac{X^d}{X^0})
\]

and assume without loss of generality that we are dealing from now on with a semimartingale \(S\).
Let $H$ be a predictable $S$-integrable process and denote by $(H \cdot S)$ the stochastic integral process of $H$ with respect to $S$, the (portfolio) wealth process. The process $H$ is called an $a$-admissible trading strategy if there is $a \geq 0$ such that $(H \cdot S)_t \geq -a$ for all $t \in [0, T]$. A strategy is called admissible if it is $a$-admissible for some $a \geq 0$. Define

$$K = \{(H \cdot S)_T : H \text{ admissible}\} \text{ and }$$

$$C = \{g \in L^\infty(P) : g \leq f \text{ for some } f \in K\}.$$ 

Then $K$ and $C$ form convex cones in $L^0(\Omega, \mathcal{F}, P)$. 
The condition *no free lunch with vanishing risk* (NFLVR) is the right concept of no arbitrage, see work of Delbaen/Schachermayer.

**Definition**

The market $\mathbf{S}$ satisfies (NFLVR) if

$$\bar{C} \cap L^\infty_+(P) = \{0\},$$

where $\bar{C}$ denotes the closure of $C$ with respect to the norm topology of $L^\infty(P)$. 
This means that a free lunch with vanishing risk exists, if there exists a free lunch $f \in L^\infty_+(P)$, which can be approximated by a sequence of portfolio wealth processes minus consumption $(f_n)$ such that $f_n \leq (H_n \cdot S)_T \in K$ with $\frac{1}{n}$-admissible integrands $H_n$, such that

$$\lim_{n \to \infty} \| f - f_n \|_\infty = 0$$

with respect to the norm topology of $L^\infty(P)$. Define the set $M_e$ of equivalent separating measures as

$$M_e = \{ Q \sim P_{\mathcal{F}_T} : E_Q[f] \leq 0 \text{ for all } f \in K \}.$$ 

If $S$ is (locally) bounded then $M_e$ consists of all equivalent probability measures such that $S$ is a (local) martingale.
Definition

Fix $Q \in \mathcal{M}_e$. Consider an extension of the original market $S' := (S, Y)$ by finitely many assets $Y$ such that the process $S'$ is a $Q$-local martingale. Consider furthermore a predictable, $S'$-integrable process $\varphi$ and the sequence of hitting times

$$\sigma_n := \inf\{ t \geq 0 : (\varphi \cdot S')_t \leq -n \}, \quad n \geq 1.$$

The trading strategy $\varphi$ is called $Q$-admissible (such as the corresponding stochastic integral, the wealth process), if

$$\liminf_{n \to \infty} E_Q[(\varphi \cdot S')_{\sigma_n} \mathbb{I}_{\{\sigma_n < \infty\}}] = 0.$$
Remark

- A trading strategy is $Q$-admissible if the price of the shortfall below levels $n$ converges to 0 for large $n$ when priced with respect to $Q$.
- The previous definition is chosen to still allow for Ansel-Stricker like conclusions.
Define

\[ L^Q = \{ x + (\varphi \cdot S') : x \in \mathbb{R}, \quad \varphi \text{ is } Q\text{-admissible} \} . \]

and

\[ L = \bigcup_{Q \in \mathcal{M}_e} L^Q . \]

We extend the set of admissible portfolios but we do not introduce arbitrages, since every wealth process \((\varphi \cdot S')\) for a \(Q\)-admissible strategy is a supermartingale. We also do not introduce free lunches, since this notion only depends on \(a\)-admissible strategies.
Now we are in the position to make our intuitive definition of numéraire portfolios precise: a numéraire portfolio is a strictly positive portfolio which allows for short-selling, i.e. the negative of its wealth process is still given by a $Q$-admissible trading strategy for some $Q \in \mathcal{M}_e$, and hence is an element of $\mathcal{L}$.

**Definition**

A strictly positive process $N \in \mathcal{L}$ with $N_0 = 1$ is called a *strong numéraire* (in discounted terms with respect to $S^0$), if

$$N \in \mathcal{L}^Q \text{ and } -N \in \mathcal{L}^Q$$

for all $Q \in \mathcal{M}_e$. It is called *weak numéraire* (in discounted terms with respect to $S^0$), if (2) holds for at least one $Q \in \mathcal{M}_e$, i.e. $N$ and $-N$ are elements of $\mathcal{L}$.
This definition has a clear economic meaning and easy consequences: as it should be, a weak numéraire qualifies as an accounting unit, where the classical change of numéraire technique is possible: there exist an equivalent measure $Q \in \mathcal{M}_e$ under which $N = (1 + (\varphi \cdot S'))$ is a true $Q$-martingale.

**Theorem**

*The following statements are equivalent:*

1. A strictly positive process $N$ with $N_0 = 1$ is a weak numéraire.
2. There exists $Q \in \mathcal{M}_e$ such that $N$ is a $Q$-martingale.
Definition

A strictly positive process $B \in \mathcal{L}$ is (modeled) in a \textit{strong bubble state} if $-B \notin \mathcal{L}$, i.e. for all $Q \in \mathcal{M}_e$ the wealth process $B$ is a strict local martingale. It is (modeled) in a \textit{weak bubble state} if $-B \notin \mathcal{L}^Q$ for some $Q \in \mathcal{M}_e$, i.e. for this $Q \in \mathcal{M}_e$ the wealth process $B$ is a strict local martingale.
Theorem

A strictly positive portfolio \( B \in \mathbf{L} \) with \( B_0 = 1 \) is in a strong bubble state if and only if \( B \) does not qualify as weak numéraire portfolio.

A strictly positive portfolio \( B \in \mathbf{L} \) with \( B_0 = 1 \) is in a weak bubble state if and only if \( B \) does not qualify as strong numéraire.
Virtual Market

Let $V \in \mathcal{L}$ with $V_0 = 1$ be a weak bubble, i.e. there is $Q \in \mathcal{M}_e$ such that $V$ is a strict $Q$-local martingale. Consider $T \in [0, T^*)$ and define the probability measure $Q^{VT}$ by

$$E_{Q^{VT}}[Y] := \frac{E_Q[YVT]}{E_Q[VT]}$$

for bounded measurable $Y$. We call the market discounted by $V$ a virtual market and its prices virtual prices with respect to $V_T$, i.e. the price of a discounted (with respect to $S^0$) $\mathcal{F}_T$-measurable claim $Y$ in this virtual market with respect to the pricing measure $Q^{VT}$ is given by

$$E_{Q^{VT}}\left[\frac{Y}{V_T}\right] = \frac{E_Q[Y]}{E_Q[V_T]}.$$
We call the difference of prices

\[ E_{Q^V_T} \left[ \frac{Y}{V_T} \right] - E_Q[Y] = E_Q[Y] \left( 1 - \frac{1}{E_Q[V_T]} \right), \quad 0 \leq T \leq T^* \] (3)

the *term structure of (il-)liquidity premia* of the numéraire \( V \) with respect to the pricing measure \( Q \).
Remark

This is also related to the notion of workable claims as introduced by Delbaen-Schachermayer and their relationship to change of numéraire theorems.
We consider the following model for a bond market. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space where the filtration satisfies the usual conditions. For each $T \in [0, \infty)$ we denote by $(P(t, T))_{0 \leq t \leq T}$ the price process of a bond with maturity $T$. For all $T$, $(P(t, T))_{0 \leq t \leq T}$ is a strictly positive càdlàg stochastic process adapted to $(\mathcal{F}_t)_{0 \leq t \leq T}$ with $P(T, T) = 1$. We assume that the price process is almost surely right continuous in the second variable, where the nullset does not depend on $t$, indeed we make

**Assumption**

There is $N \in \mathcal{F}$ with $P(N) = 0$ such that

$$N \supseteq \bigcup_{t \in [0, \infty)} \{ \omega : T \rightarrow P(t, T)(\omega) \text{ is not right continuous} \}.$$
Assumption

We make the following assumption on uniform local boundedness for $P(., T)$ and local boundedness for $P(., T)^{-1}$:

1) **For any** $T$ **there is** $\varepsilon > 0$, **an increasing sequence of stopping times** $\tau_n \to \infty$ **and** $\kappa_n \in [0, \infty)$ **such that**

$$P(t, U)^{\tau_n} \leq \kappa_n,$$

**for all** $U \in [T, T + \varepsilon)$ **and all** $t \leq T$.

2) **There exists a nonempty set** $T \subset [0, \infty)$ **such that**

$$\left(\frac{1}{P(t, T^*)}\right)_{0\leq t \leq T^*}$$ **is locally bounded for all** $T^* \in T$. 
In the following assumption we consider a numéraire related to a *terminal maturity* $T^* \in \mathcal{T}$.

**Assumption**

*For all finite collections of maturities $T_1 < T_2 < \cdots < T_n \leq T^*$ with $T^* \in \mathcal{T}$ there exists a measure $Q \sim P|_{\mathcal{F}_{T^*}}$ such that*

\[
\left( \frac{P(t, T_i)}{P(t, T^*)} \right)_{0 \leq t \leq T_i}
\]

*is a local $Q$-martingale, $i = 1, \ldots, n$.*

The measure $Q$ from Assumption 3 is called the $T^*$-forward-measure for the finite market consisting of bonds $P(., T_i)$, $i = 1, \ldots, n$ and the numéraire $P(., T^*)$. 
Note that we do not assume the existence of a short rate or even a bank account. Moreover, we do not assume that \( P(., T) \) is a semimartingale. However, Assumption (3) implies that, for a finite collection of maturities, only bonds in terms of the numéraire \( P(., T_n) \) are semimartingales under the objective measure \( P \), because they are local martingales under the equivalent measure \( Q \). Moreover, they are locally bounded because we assumed that \( P(., T) \) is locally bounded, for any \( T \), and \( P(., T^*)^{-1} \) is locally bounded for \( T^* \in \mathcal{T} \).
Assumption (3) means that for a finite selection of bonds considered with respect to a certain numéraire (the bond with the largest maturity) there exists an equivalent local martingale measure. Our aim will be the following: for a fixed maturity $T^* \in \mathcal{T}$, we aim at finding a measure $Q^*$ such that all bonds with maturity $T \leq T^*$ are local martingales under $Q^*$ in terms of the numéraire $P(t, T^*)$. 
We choose a finite time horizon $T > 0$ as this will be sufficient for our purpose. Let $(S^n_t)_{t \in [0,T]}$, $n = 1, 2, \ldots$, be a sequence of semimartingales, where $S^n$ takes values in $\mathbb{R}^{d(n)}$, based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ where the filtration satisfies the usual assumptions. For each $n \geq 1$ we define a classical market model (referred to as finite market $n$) given by the $\mathbb{R}^{d(n)}$–valued semimartingale $S^n$ (which describes the price processes of $d(n)$ tradable assets). We assume that the assets are already discounted with respect to one numéraire, so we have that one of the $d(n)$ assets equals 1. For our purposes it is sufficient to assume that there is a sequence $S^i$, $i = 0, 1, \ldots$ of semimartingales, such that $S^0_t \equiv 1$ and such that $(S^n_t) = (S^0_t, S^1_t, \ldots, S^n_t)$. In this case $d(n) = n + 1$. 
Let $H$ be a predictable $S^n$-integrable process and, as previously, $(H \cdot S^n)_t$ the stochastic integral of $H$ with respect to $S$. The process $H$ is an admissible trading strategy if $H_0 = 0$ and there is $a > 0$ such that $(H \cdot S^n)_t \geq -a$, $0 \leq t \leq T$. Define

$$K^n = \{(H \cdot S^n)_T : H \text{ admissible}\} \text{ and } C^n = (K^n - L_+^0) \cap L^\infty.$$ (4)

$K^n$ can be interpreted as the cone of all replicable claims in the finite market $n$, and $C^n$ is the cone of all claims in $L^\infty$ that can be superreplicated in this market.
Define the set $\mathbf{M}_e^n$ of equivalent separating measures for the finite market $n$ as

$$\mathbf{M}_e^n = \{ Q \sim P : E_Q[f] \leq 0 \text{ for all } f \in \mathcal{C}^n \}$$

$$= \{ Q \sim P : E_Q[f] \leq 0 \text{ for all } f \in \mathcal{K}^n \}. \quad (5)$$

If $\mathcal{S}^n$ is (locally) bounded then $\mathbf{M}_e^n$ consists of all equivalent probability measures such that $\mathcal{S}^n$ is a (local) martingale.
A large financial market is the sequence of the finite market models $n$, i.e. the sequence of the market models induced by the $d(n)$-dimensional semimartingales $S^n$. As a consequence, we cannot trade with an actually infinite number of securities (so that we avoid artificially introduced infinite-dimensional trading strategies), but we can trade in more and more assets and in this way approximate something infinite-dimensional.
We impose the following assumption, which is standard in the theory of large financial markets:

\[ M^n_e \neq \emptyset, \quad \text{for all } n \in \mathbb{N}. \] (6)

This implies that any no arbitrage condition (such as \textit{no arbitrage}, \textit{no free lunch with vanishing risk}, \textit{no free lunch}) holds for each finite market \( n \).
No asymptotic free lunch

(NAFL) is the large financial markets analogue of the classical no free lunch condition (NFL) of Kreps. We will first recall the classical NFL condition here for a finite market $n$. Let $C^n$ be defined as in (4).
Definition

The condition NFL holds on the finite market $n$ if

$$\overline{C^n}^* \cap L^\infty = \{0\},$$

where $\overline{C^n}^*$ denotes the weak-star-closure of $C^n$. (7)
This means by superreplicating claims in an admissible way with a finite number of assets we cannot approximate in a weak-star sense a strictly positive gain.

Now NAFL can be defined in analogous way as the condition NFL but for the whole sequence of sets $C^n$:

**Definition**

A given large financial market satisfies NAFL if

$$\bigcup_{n=1}^{\infty} C^n \cap L^\infty = \{0\}.$$
If NAFL holds then it is not possible to approximate a strictly positive profit in a weak-star sense by trading in any finite number of the given assets (although we can use more and more of them).

Note that in the literature the term large financial market is used for a more general concept where each market $n$ is based on a different filtered probability space. So, in our setting, we will not have to deal with technicalities which are common in large financial markets.
Definition

Let $T^* \in \mathcal{T}$ where $\mathcal{T}$ is the set from Assumption 2. Fix a sequence $(T_i)_{i \in \mathbb{N}}$ in $[0, T^*]$. Define the $n + 1$-dimensional stochastic process $(S^n) = (S^0, \ldots, S^n)$ on $[0, T^*]$ as follows:

$$S^i_t = \begin{cases} \frac{P(t, T_i)}{P(t, T^*)} & \text{for } 0 \leq t \leq T_i \\ \frac{1}{P(T_i, T^*)} & \text{for } T_i < t \leq T^* \end{cases},$$  

for $i = 1, \ldots, n$ and $S^0_t = \frac{P(t, T^*)}{P(t, T^*)} \equiv 1$. The large financial market consists of the sequence of classical market models given by the $(n + 1)$-dimensional stochastic processes $(S^n)_{t \in [0, T^*]}$ based on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T^*]}, P|_{\mathcal{F}_{T^*}})$. 


Definition

The bond market \( (P(t, T))_{0 \leq t \leq T} \) for \( 0 \leq T \leq T^* \) is said to satisfy NAFL if there exists a dense sequence \((T_i)_{i \in \mathbb{N}}\) in \([0, T^*]\), such that the large financial market of Definition 9 satisfies the condition NAFL.
Since all involved semimartingales $S^n$ are locally bounded due to Assumption 2, it is sufficient to deal with equivalent local martingale measures. Hence, the set $M^n_e$ from (5) is given as follows:

$$M^n_e = \{ Q^n \sim P _{|F_T} : S^n \text{ local } Q^n\text{-martingale} \}.$$ 

By Assumption 3 we have that $M^n_e \neq \emptyset$ for all $n \in \mathbb{N}$, so the standard assumption (6) for large financial markets holds. Note that this also implies that each $S^n$ is a semimartingale, so this is not a problem in Definition 9.
Theorem

Fix any $T^* \in \mathcal{T}$ and let Assumption 1, 2 and Assumption 3 hold. Then, the bond market satisfies NAFL, if and only if there exists a measure $Q^* \sim P|_{\mathcal{F}_{T^*}}$ such that $\left( \frac{P(t, T)}{P(t, T^*)} \right)_{0 \leq t \leq T}$ is a local $Q^*$-martingale for all $T \in [0, T^*]$. 
In Theorem 11 we consider the NAFL condition for the large financial market as in Definition 9 with respect to one fixed, dense sequence of $T_i$ in $[0, T^*]$. However, as there is a local martingale measure for all bond prices discounted by the numéraire, the general theorem about NAFL in large financial markets implies that for any large financial market (induced by the bond market via any sequence of maturities) NAFL holds.
It is possible to obtain a candidate process for the bank account by a limit of rolled over bonds as we show now. Throughout this section we assume that all the assumptions of Theorem 11 hold.

**Definition**

Let \( 0 = t^n_0 < t^n_1 < \cdots < t^n_{k_n} = T^* \) be a sequence of refining partitions of \([0, T^*]\). Define, for each \( n \), the *roll-over* \( B^n \) as follows: \( B^n_0 = 1 \) and

\[
B^n_t = \begin{cases} 
\prod_{i=1}^{j} \frac{1}{P(t^n_{i-1}, t^n_i)} & \text{for } t = t^n_j, j = 1, \ldots, k_n \\
B^n_{t_j} P(t, t^n_j) & \text{for } t^n_{j-1} < t \leq t^n_j, j = 1, \ldots, k_n
\end{cases}
\]
Lemma

There exists a self-financing strategy $\hat{H}_t^n = (\hat{H}_t^1, \ldots, \hat{H}_t^{k_n})$ on the market containing the $k_n$-dimensional asset $\hat{S}^n(\cdot) = (P(., t^n_1), \ldots, P(., t^n_{k_n}))$ such that $B_t^n = \langle \hat{H}_t, \hat{S}_t \rangle$. Discounted by the numéraire $P(t, t^n_{k_n}) = P(t, T^*)$ this gives an admissible strategy $H^n$ such that

\[
\frac{B_t^n}{P(t, T^*)} = \frac{1}{P(0, T^*)} + (H^n \cdot S^n)_t > 0,
\]

where $S^n$ is the process $\hat{S}^n$ discounted by the numéraire $P(t, T^*)$. In particular, $\left(\frac{B_t^n}{P(t, T^*)}\right)_{0 \leq t \leq T^*}$ is a positive local martingale and hence a supermartingale with respect to the measure $Q^*$ of Theorem 11.
Theorem

Let \( ((B^n_t)_{0 \leq t \leq T^*}) \) be the sequence of roll-overs given as in Definition 12. There exists a sequence of convex combinations \( \tilde{B}^n \in \text{conv}(B^n, B^{n+1}, \ldots) \) and a càdlàg stochastic process \( (B_t)_{0 \leq t \leq T^*} \), henceforward called generalized bank account, such that

\[
B_t = \lim_{q \downarrow t} \lim_{n \to \infty} \tilde{B}^n_q,
\]

with \( B_0 \leq 1 \) and \( 0 \leq B_t < \infty \), for all \( t \leq T^* \). The generalized bank account has the following properties.

1. The process \( (V_t)_{0 \leq t \leq T^*} \), where \( V_t = \frac{B_t}{P(t, T^*)} \), is a supermartingale with respect to the measure \( Q^* \) of Theorem 11.

2. If \( 0 < P(t, T) \leq 1 \), for all \( T \leq T^* \), then \( P(B_t \geq 1) = 1 \), for all \( t \leq T^* \).
With a view what it means to be a numéraire we can ask why just terminal bonds qualify as numéraires by default in our setting: the answer is that we could take any other reasonably behaved stochastic process (the inverse has to be locally bounded) and plug it into Assumption 3 instead of $P(., T^*)$. Conclusions would remain the same, of course with a different meaning on what we would call numéraire in this setting. For instance we could think of taking discrete roll-over bonds as numéraires if we want to claim that this portfolio can be shortened.
Conclusion

- Analysis of bond markets beyond the paradigm of an existing short rate via large financial markets.
- Existence proof for short rates, but the bank account process can not necessarily be chosen as numéraire.
- Introduction of a dichotomy between bubbles and numéraires.